Stably Placing Piecewise Smooth Objects

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1 Introduction

Given a piecewise smooth object K, we study how it may be placed stably on a horizontal and flat surface. We consider two versions of this problem.

The static version seeks to determine the set of *stable poses* of K, or the local minima of its gravitational potential energy as a function of its orientation with respect to the direction of the gravity. Such information is useful for example, in a model-based vision system, where the complexity of the recognition of static objects can be reduced as a result of reducing the number of unknown parameters by two.

The more general version seeks to determine not only the stable poses of K, but also a *capture region* for each stable pose $\mathbf{p}$, namely the set of all initial poses released from which K will eventually roll into $\mathbf{p}$. Consider the last step of the path-planning algorithm for a robot where the robot hand is about to place an industrial part on a flat surface. Knowledge of the capture region for the desired object pose (which must be stable) helps the path-planning algorithm determine the required precision of this operation. Comparison among the capture regions associated with various stable poses also leads to a better decision if the choice of the object pose is an option for the algorithm.

Kriegman [Kri91] has taken the direct approach to analyzing the stable pose problem. A complete implementation using this method would have to include sixteen different cases. His implementation, using homotopy method in solving the systems of algebraic equations, demonstrate the feasibility of the algorithm by dealing with objects composed of natural quadratic surface patches cut by planes without vertices. ‡ The computation time, however, is not very attractive even for a simple object such as the one shown in Figure 1. (This is our reconstruction of an example taken from [Kri91].) In extending the work to capture regions, Kriegman [Kri95] gives a quadratic time algorithm for polyhedral objects. His direct approach using stratified Morse theory in analyzing the capture regions for an object composed of algebraic surface patches, however, has to make certain non-singularity assumptions that may be difficult to ensure in practice.

In this paper we take the dual approach to solve these two problems and extend Kriegman's results in several ways. We establish combinatorial upper bounds for the complexity of the potential energy function, from which the results on the stable poses and capture regions can be derived. This approach also allows us to remove the non-singularity constraints and to enumerate the degenerate cases in a systematic way. Furthermore, the algebraic overhead is greatly reduced when K consists solely of quadratic surface patches cut by planes.

2 Object Model and Assumptions

We assume that a piecewise smooth object K is described as an unordered set of N (open) faces $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ whose union constitute a topological polyhedron embedded in E^3. [Cai68] Thus K is assumed to be compact but may have an arbitrary genus, need not be connected, and need not be a manifold. Furthermore, each 2-face is assumed to be a G^2 simply-connected surface patch, and each 1-face a G^2 curve segment. In order to give explicit formulas for the following results, we also assume that each face is a regular portion of an algebraic variety although this is not a requirement for the validity of the cell theorem. Given a face φ, we use *host* to refer to the entirety of the algebraic variety of which φ is a portion.

Finally, we assume that the coordinate system is so chosen that the origin coincides with the center of mass of K. We will denote the origin by $\mathcal{O}$.

3 Duality

The *pedal* of a surface is the loci of the perpendicular foot dropped from $\mathcal{O}$ to a tangent plane to the surface as the point of tangency moves on the surface. [BG92] We naturally define $\texttt{ped}\,\varphi$, the pedal of a 2-face φ to

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‡Cases involving vertices are explicitly left out from the enumeration in his paper. They do not seem to pose any more theoretical or practical difficulties than the enumerated cases since the algebra for the former is simpler.

be the portion of its host's pedal which corresponds to taking the tangent planes only within the extent of φ. We extend this notion and define the pedal of a 1-face φ to be the collection of the perpendicular feet from $\mathcal{O}$ to the planes which have a contact of order at least one with φ. Finally we define the pedal of a vertex to be the collection of such perpendicular feet to the planes incident on the vertex.

The inversion of a point $\mathbf{p}$ in space is the the point $\mathbf{p}/||\mathbf{p}||^2$. We define the dual of a face φ to be the pointwise inverted image of its pedal and denote it by $\tilde{\varphi} = \mathtt{inv}\,\mathtt{ped}\,\varphi$. It can be shown that

Theorem 1 *In the arrangement of the duals of K's faces, the cell which contains the origin is compact and convex, and it is the polar set of K.*

The polar set $K^* \stackrel{\text{def}}{\equiv} \{\mathbf{x} : \mathbf{x}\cdot\mathbf{y} \leq 1\}$ of a set K is a classical form of duality. [Lay92]

For objects consisting of algebraic 2-faces and edges, the dual faces can be computed by performing variable elimination. [Col75, IK93, Man93] We shall see, however, that this problem reduces to linear algebra on small matrices (4 by 4) when K consists of quadratic surface patches cut by planes, thus making the dual approach very attractive in practical terms.

4 The Potential Energy Function

Consider representing the boundary of the polar set K^* in spherical coordinates. The radius is thus a single-valued function of the longitude and latitude since K^* is convex. The graph of the radius function of the polar set is precisely the reciprocal of the potential energy of the object as a function of its pose. Several questions regarding the stable poses and their respective capture regions can be answered once we construct a description of this function. In particular, the stable poses correspond to the local maxima of this function. It is shown in [Hun95] that

Theorem 2 *Let K be a piecewise smooth object with N faces such that each face has a bounded number of subfaces. Then the boundary of K^* can be constructed in randomized expected time $O(N^{2+\epsilon})$ for arbitrary but fixed $\epsilon > 0$.*

The proof is by way of reduction to Sharir's algorithm [Sha93] for constructing the lower envelope in an arrangement of surface patches. Thus we have the following

Corollary 1 *There are at most $K^{2+\epsilon}$ stable poses for the object in Theorem 2. It can be computed in the same amount of time, barring algebraic overhead.*

Corollary 2 *The capture regions of the object in Theorem 2 have a structural complexity of $O(N^{4+\epsilon})$ and can be computed in deterministic time $O(N^{4+\epsilon})$.*

Moreover, it is possible to treat in a systematic way the degenerate cases and coincidences, which are hard even to enumerate using the direct approach. Consider three identical fat ellipsoids stacked on their "flat" sides as in Figure 2. Aside from the two stable poses corresponding to two one-point contacts (top and bottom), it is not hard to see that there are two other stable poses corresponding to the two contacts where each ellipsoid contribute a point and where the three points of contact line up. It is not obvious whether the latter stable poses should be considered one-point, two-point, or three-point contacts. When one looks at its potential energy function in the dual space, one realizes that in the latter case each local maxima is the intersection of three saddles, one of which can be considered gratuitous in forming the local maximum (Figure 3). In general such special cases can be resolved in the algorithm by examining only the local structure of the radius function as opposed to having to consider features that are far apart in the primal space. Distinguishing the local maxima among all "critical poses" where certain partial derivatives vanish is similarly facilitated by working in the dual space.

5 Quadratic Surface Patches Cut by Planes

The results in the previous section are more of theoretical interest since we have focused only on the combinatorial complexity. Kriegman's experimental results, however, show that the dominating factor in computing the stable poses is the algebraic overhead. Taking the direct approach, his implementation has to solve a system of quadratic functions in up to nine variables for simple parts made of cylinder sections and spheres. Fortunately the dual approach also reduces the algebraic complexity dramatically in the case of quadratic surface patches cut by planes. The variable elimination in the dualizing step reduces to simple linear algebra operations.

In the special case of quadratic surfaces, the inverted pedal becomes the classical duality in projective geometry with a slight modification (in fact just reflect every vector with respect to the origin). Thus to compute the dual of a quadratic face φ, we simply represent the host surface by a 4 by 4 real symmetric matrix, say Q and then invert Q — provided that the surface is non-degenerate and hence Q is non-singular. The boundary of the dual surface can also be obtained by mere linear algebra without going through the elimination process. Let $\mathbf{y}$ be the 4-vector representation of a cutting plane H of φ. The pole of H with respect to the host of φ dualizes to the cutting plane of $\tilde{\varphi}$ which determines the extent of the dual face, and is simply $\mathbf{y}Q^{-1}$.

We also have to give a recipe for computing the duals in the degenerate cases such as cones and cylinders. In this case we first compute $Q = U\Lambda U^T$ where U is orthogonal and Λ is diagonal. Let Λ' be the same as Λ except with the vanishing diagonal entry replaced by 1. Then $\tilde{\varphi}$ is the intersection of $U(\Lambda')^{-1}U^T$ with the plane whose coefficients are the eigenvector of Q corresponding to the vanishing eigenvalue.

Finally the duals of the edges can also be computed efficiently. Let γ be an edge whose host is the intersection of a non-degenerate quadratic hypersurface represented by Q with a plane represented by the 4 by 1 matrix N. We first perform a QR factorization and write

$$N = (U_1 U_2) \begin{pmatrix} R \\ 0 \end{pmatrix}$$

where U_1 is 4 by 1, U_2 is 4 by 3, both orthogonal, and where R is upper-triangular and non-singular. Next write Q as

$$Q = (U_1 U_2) \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix}.$$

Then it can be shown that $\tilde{\gamma}$ is the (degenerate) quadratic surface $U_2 C^{-1} U_2$.

Putting all this together, we conclude that the dominant algebraic calculations involved in finding the stable poses of such objects are:

- Solving a system of 3 quadratic equations for each vertex of $\partial(K^*)$.
- Maximizing a quadratic function over the intersection of two quadratic surfaces for each edge of $\partial(K^*)$. Note that the intersection has a closed form parameterization.
- Maximizing a quadratic function over a quadratic surface for each dual face. This can be reduced to finding the root of a univariate polynomial of degree 6.

Note that this does not take into account the reduction to the lower envelope problem and its solution, which would incur additional algebraic overhead. In the next section we propose an approximation algorithm that may not be asymptotically optimal, but is extremely efficient in practical terms, as demonstrated by our implementation.

6 An Approximation Algorithm

In the dual space, consider shooting rays from the origin with nearly regular spacing between the rays, for example, doing so uniformly at the grid points of the longitudinal and latitudinal lines on the unit sphere. Create a planar graph whose vertices correspond to the rays and whose edges connect the neighboring rays in a natural way (such as the longitudinal sections and latitudinal sections in our example). For each ray find the first dual surface it hits and compute the tangent plane at the point of intersection. The planar graph can be refined by adding vertices between adjacent vertices which fall on different dual faces, adding edges between the newly added vertices which surround the same face of the graph and fall on the same dual edge, and finally adding new vertices which correspond to the intersections of the new edges. In this way we create a finer description of the potential energy function than is necessary. The combinatorial complexity is raised by a quadratic factor of the resolution of the rays (and reduced by a linear factor in the number of the object features) whereas the algebraic overhead required to construct it is substantially lower. Our partial implementation using `C++` shows that the performance is rather realistic.

It takes roughly one minute to create, among other things, a mesh of about 100 by 50 (which is sufficient for this particular example) for the radius function of the polar set of a 6-piece pipe like Figure 1 (reconstructed from [Kri91]) even on an i486-33 running OS/2 or Linux. The radius function of the polar set is shown in Figure 4.

Such uniform sampling scheme in the dual space can be understood as weighing the sampling density (number of rays per unit area on a face), *in the primal space*, by the Gaussian curvature of the face since the latter is the limiting ratio between the area of the Gaussian sphere and the surface area in the primal space. Intuitively, we are sampling more frequently at badly curved places, and saving our "sampling bandwidth" at relatively area of the convex hull of the original object.

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References

[BG92] J. W. Bruce and P. J. Giblin. *Curves and Singularities.* Cambridge University Press, 2nd edition, 1992.

[Cai68] Stewart Scott Cairns. *Introductory Topology.* Ronald Press Company, 1968.

[Col75] G. E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In *Second GI Conference on Automata Theory and Formal Languages*, volume 33, pages 134–183. Srpinger-Verlag, 1975.

[Hun95] Chao-Kuei Hung. *Convex Hull of Surface Patches: Construction and Applications.* PhD

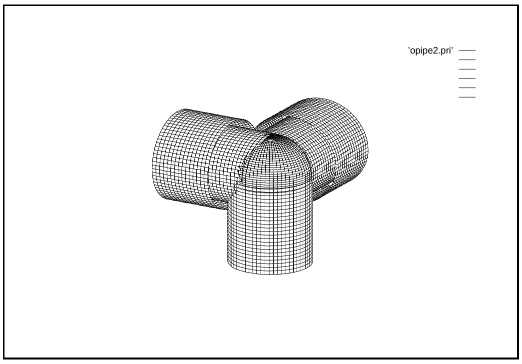

Figure 1: A PVC pipe fitting. The center of mass is just outside the sphere in the octant wedged by the three branches.

thesis, Computer Science Department, University of Southern California, 1995.

[IK93] Doug Ierardi and Dexter Kozen. Parallel resultant computation. In John H. Reif, editor, *Synthesis of Parallel Algorithms*. Morgan Kaufmann, 1993.

[Kri91] David J. Kriegman. Computing stable poses of piecewise smooth objects. In *Computer Vision Graphics Image Processing: Image Understanding*, 1991.

[Kri95] David J. Kriegman. Let them fall where they may: Capture regions of curved objects and polyhedra. Technical Report 9508, Yale University, June 1995. Submitted to the International Journal of Robotics Research.

[Lay92] Steven R. Lay. *Convex Sets and Their Applications*. John Wiley & Sons, Inc., 1992. original edition 1982.

[Man93] Dinesh Manocha. Solving polynomial systems for curves, surface, and solid modeling. In *Proc. of ACM/SIGGRAPH*, 1993.

[Sha93] Micha Sharir. Almost tight upper bounds for lower envelopes in higher dimensions. In *Symposium on Foundations of Computer Science*, pages 498–507, 1993.

[SS85] Otto Schreier and Emanuel Sperner. *Projective Geometry of n Dimensions*. Chelsea Publishing Company, New York, 1985.

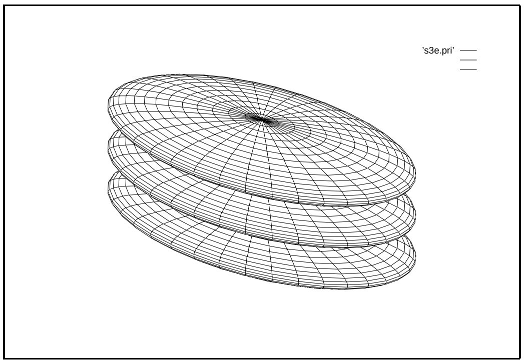

Figure 2: Is this a one-point, two-point, or three-point contact?

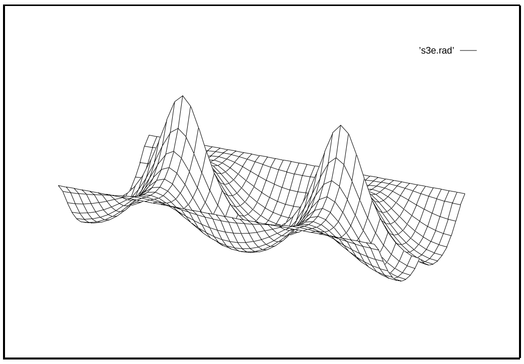

Figure 3: The radius function of the stacked ellipsoids.

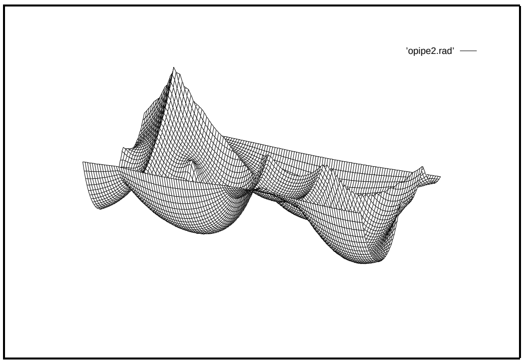

Figure 4: The radius function of the polar set of the pipe fitting.

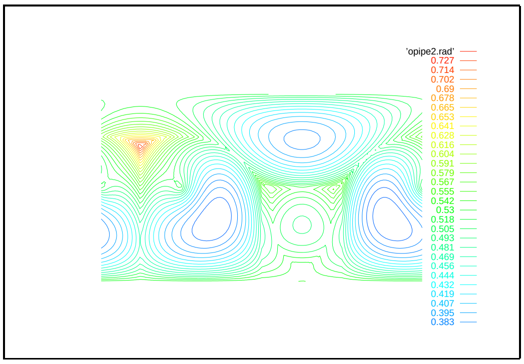

Figure 5: The contour plot of the radius function of the polar set of the pipe fitting.